MAT 219 (lecture 6)

§7.4 Basic Theory of Systems
§7.3 Eigenvalues & Eigenvectors

Note: These lectures will not follow the book precisely.

Recall: A system of linear 1^{st} order equations is

$$\frac{dx_1}{dt} = p_{11}(t)\,x_1 + p_{12}(t)\,x_2 + \cdots + g_1(t)$$

$$\frac{dx_2}{dt} = p_{21}(t)\,x_2 + p_{22}(t)\,x_2 + \cdots + g_2(t)$$

$$\vdots$$

where x_1, x_2 etc are unknown functions of t.

$\rightarrow$ **Solution** is functions for x_1, x_2 etc that make equations true.

Note: We will focus on systems with constant coefficients

$\rightsquigarrow p_{ij}(t) =$ numbers, not functions!

EX:

$$x_1' = 2x_1 - x_2 + e^t$$
$$x_2' = 3x_1 - 2x_2 - e^t$$

has solution

$$x_1 = \boxed{c_1}e^t + \boxed{c_2}e^{-t} + e^t + 2te^t$$
$$x_2 = \boxed{c_1}e^t + 3\boxed{c_2}e^{-t} + 2te^t$$

Check:

$$x_1' = (c_1e^t + c_2e^{-t} + e^t + 2te^t)'$$
$$= c_1e^t - c_2e^{-t} + \underbrace{e^t + 2e^t}_{3e^t} + 2te^t$$

$$2x_1 - x_2 + e^t = 2(c_1e^t + c_2e^{-t} + e^t + 2te^t) - (c_1e^t + 3c_2e^{-t} + 2te^t) + e^t$$

$$= c_1e^t - c_2e^t + 3e^t + 2te^t$$

(skip checking other equation)

Look at phase plane to understand solutions — graph x_1 vs x_2

EX

$$x_1' = 2x_1 - x_2$$
$$x_2' = 3x_1 - 2x_2$$

For functions this is like combining

$$\begin{matrix} x_1(t) \\ x_2(t) \end{matrix} \longmapsto \underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

vector function (curve in $\mathbb{R}^2$)

system becomes:

$$\frac{d\underline{\underline{x}}}{dt} = \underbrace{\begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots \\ p_{21}(t) & p_{22}(t) & \cdots \\ \vdots & & \end{bmatrix}}_{\underline{\underline{P}}(t)} \underline{\underline{x}} + \underbrace{\begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \end{bmatrix}}_{\underline{\underline{g}}(t)}$$

"matrix function"

with solution

$$\underline{\underline{x}} = c_1 \underline{\underline{x}}^{(1)} + c_2 \underline{\underline{x}}^{(2)} + \cdots + \underline{\underline{x}}_p$$

EX

$$\underline{\underline{x}}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \underline{\underline{x}} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$$

has solution

$$\underline{\underline{x}} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} t e^t$$

EX

$$x_1' = x_1 + x_2 + e^{-2t}$$
$$x_2' = 4x_1 - 2x_2 - 2e^t \quad \rightsquigarrow \quad \underline{\underline{x}}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \underline{\underline{x}} + \begin{bmatrix} e^{-2t} \\ -2e^t \end{bmatrix}$$

soln

$$x_1 = c_1 e^{-3t} + c_2 e^{2t} + \tfrac{1}{2} e^t$$
$$x_2 = -4c_1 e^{-3t} + c_2 e^{2t} - e^{-2t}$$

$$\Downarrow$$

$$\underline{\underline{x}} = c_1 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} \tfrac{1}{2} e^t \\ -e^{-2t} \end{bmatrix}$$

these are the equilibrium solutions (the straight lines in phase plane)

---

§7.4 Basic Theory

Linear system of DE is

$$\boxed{\underline{\underline{x}}' = \underline{\underline{P}}(t)\underline{\underline{x}} + \underline{\underline{g}}(t) \quad \text{w/ Initial values} \quad \underline{\underline{x}}(t_0) = \underline{\underline{x}}_0}$$

EX

$$\underline{\underline{x}}' = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \underline{\underline{x}} + \begin{bmatrix} 8e^t \\ -e^t \end{bmatrix} \text{ with } \underline{\underline{x}}(0) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

(work)

General solution:

$$\underline{\underline{x}} = c_1 \begin{bmatrix} 4 \\ -1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^t$$

Initial value:

$$\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \underline{\underline{x}}(0) = c_1 \begin{bmatrix} 4 \\ -1 \end{bmatrix} \cancel{e^{0}} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cancel{e^{0}} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \cancel{e^{0}}$$

... solve for c_1 & c_2

General solution to

$$x' = Px + g$$

has two parts

$$x = \underbrace{c_1 x^{(1)} + \cdots}_{\text{part with constants}} + \underbrace{x_p}_{\substack{\text{part without}\\ \text{constants}}}$$

EX $x' = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 8 \\ -1 \end{bmatrix} e^t$

has soln

$$x = \underbrace{c_1 \begin{bmatrix} 4 \\ -1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}}_{\text{part with constants}} + \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix} e^t}_{\substack{\text{Part}\\ \text{without}\\ \text{const.}}}$$

FACT: The part of solution with constants is solution to

$$x' = Px \quad (\text{no } g)$$

"Associated Homogeneous System"

Plan: (1) Find part of general solution with constants.

look at $x' = Px + \cancel{g}$

(2) Find the other part.

FACT: If $x^{(1)}$ and $x^{(2)}$ are solutions to $x' = Px$ (Homogeneous) then so is $c_1 x^{(1)} + c_2 x^{(2)}$

Plan: To find part of general solution with constants

(1) look for a few different solns

(2) multiply by constants and add.

Q: How many different solutions do we need?

→ How do we know when solutions are different enough?

FACT: If $x' = Px$ has size n (n equations) then solution should have n constants.

→ Initial value $x(t_0) = x_0$ gives n-values. (which should determine n constants.)

FACT: Solutions $x^{(1)}, x^{(2)}, \ldots$ are different enough if

$$0 \neq \text{Wronskian} = \det \begin{bmatrix} | & | & \\ x^{(1)} & x^{(2)} & \cdots \\ | & | & \end{bmatrix}$$

(explain later)

Plan: To solve $\underline{\underline{x}}' = \underline{\underline{P}}\,\underline{\underline{x}} + \underline{g}$

(1) Look at $\underline{\underline{x}}' = \underline{\underline{P}}\,\underline{\underline{x}}$

→ Find n different (simple) solutions

→ homogeneous solution is

$$\underline{\underline{x}} = c_1 \underline{\underline{x}}^{(1)} + c_2 \underline{\underline{x}}^{(2)} + \dots + c_n \underline{\underline{x}}^{(n)}$$

(2) Look at $\underline{\underline{x}}' = \underline{\underline{P}}\,\underline{\underline{x}} + \underline{g}$

→ Do something clever
(Variation of Parameters)

(3) ??

(4) Profit !!

§7.5 (With bits of §7.3 mixed in)

Homogeneous Linear Systems
w/ Constant Coefficients.

⇝ all p_{ij} are numbers $p_{ij}(t) = a_{ij}$

$$x_1' = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$$
$$x_2' = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n$$
$$\vdots$$

As a vector DE this is

$$\underline{\underline{x}}' = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \\ \vdots & & \end{bmatrix}}_{A} \underline{\underline{x}}$$

EX

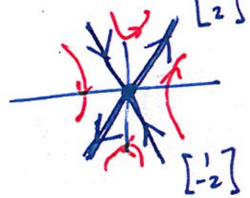
$$x_1' = x_1 + x_2$$
$$x_2' = 4x_1 + x_2$$
$$\rightsquigarrow \underline{\underline{x}}' = \underbrace{\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}}_{A} \underline{\underline{x}}$$

EX

$$x_1' = x_1 + x_2 + 2x_3$$
$$x_2' = x_1 + 2x_2 + x_3$$
$$x_3' = 2x_1 + x_2 + x_3$$
$$\rightsquigarrow \underline{\underline{x}}' = \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}}_{A} \underline{\underline{x}}$$

→ Looking for a few good solutions....

EX $\underline{\underline{x}}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \underline{\underline{x}}$ ⇝ phase plane



→ on line $\begin{bmatrix} 1 \\ 2 \end{bmatrix} t$ going outward.

→ on line $\begin{bmatrix} 1 \\ -2 \end{bmatrix} t$ going inward.

Why? At points of form $\begin{bmatrix} 1 \\ 2 \end{bmatrix} t$

$$\underline{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} t = \begin{bmatrix} 1+2 \\ 4+2 \end{bmatrix} t$$
$$= \begin{bmatrix} 3 \\ 6 \end{bmatrix} t = \begin{bmatrix} 1 \\ 2 \end{bmatrix} 3t$$

⟹ on this line system looks like $x' = 3x$ ⇝ $\boxed{\text{solution } x = ce^{3t}}$

$\Rightarrow$ one solution to system is

$$\underline{x}^{(1)} = \begin{bmatrix}1\\2\end{bmatrix} e^{3t}$$

At points like $\begin{bmatrix}1\\-2\end{bmatrix}t$ get

$$\underline{x}' = \begin{bmatrix}1 & 1\\4 & 1\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix}t = \begin{bmatrix}1-2\\4-2\end{bmatrix}t$$

$$= \begin{bmatrix}-1\\2\end{bmatrix}t = \begin{bmatrix}1\\-2\end{bmatrix}(-1)t$$

$\Rightarrow$ another solution to system is

$$\underline{x}^{(2)} = \begin{bmatrix}1\\-2\end{bmatrix} e^{-t}$$

General solution:

$$\underline{x} = c_1\begin{bmatrix}1\\2\end{bmatrix}e^{3t} + c_2\begin{bmatrix}1\\-2\end{bmatrix}e^{-t}$$

If we can find these special lines then we will know the answer.

EX $\underline{x}' = \begin{bmatrix}1 & 1 & 2\\1 & 2 & 1\\2 & 1 & 1\end{bmatrix}\underline{x}$

Special lines are $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\0\\1\end{bmatrix}$, $\begin{bmatrix}1\\-2\\1\end{bmatrix}$

$\Downarrow$ 4, $\Downarrow$ -1, $\Downarrow$ 1

Solutions: $\begin{bmatrix}1\\1\\1\end{bmatrix}e^{4t}$, $\begin{bmatrix}-1\\0\\1\end{bmatrix}e^{-t}$, $\begin{bmatrix}1\\-2\\1\end{bmatrix}e^{t}$

$\Rightarrow$ How do we find the special lines?

$$A\underline{v} = \lambda\underline{v} \qquad \left.\begin{array}{l}\underline{v}: \text{"Eigenvectors"}\\ \lambda: \text{"Eigenvalues"}\end{array}\right\} \S\, 7.3$$

(Remark: Computers find eigenvectors first, then eigenvalues (like we did)

$\Rightarrow$ Humans ~~shoot first & ask questions later~~ find eigenvalues first & then get eigenvectors)

EX: Find eigenvalues / eigenvectors of

$$A = \begin{bmatrix}1 & 1\\4 & 1\end{bmatrix}$$

Want $\begin{bmatrix}1 & 1\\4 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \lambda\begin{bmatrix}x\\y\end{bmatrix}$

i.e. $\begin{cases} x + y = \lambda x\\ 4x + y = \lambda y\end{cases}$

$$-\begin{cases}\big((1-\lambda)x + y = 0\big)(1-\lambda)\\ 4x + (1-\lambda)y = 0\end{cases}$$

FINISH NEXT TIME

$$(1-\lambda)^2 x - 4x = 0$$

$$(1 - 2\lambda + \lambda^2 - 4)x = 0$$

$$(\lambda^2 - 2\lambda - 3)x = 0$$

Note: We need a solution with $x \neq 0$

so $\underbrace{(\lambda^2 - 2\lambda - 3) = 0}_{\text{characteristic equation}}$

Since $x \neq 0$ must have

$$\lambda^2 - 2\lambda - 3 = 0 \leftarrow \text{"characteristic equation"}$$

$$(\lambda - 3)(\lambda + 1) = 0$$

Eigenvalues: $\lambda = 3, -1$

Note: These are rates of growth in solutions to $\underline{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \underline{x}$

$\lambda = 3 \rightsquigarrow e^{3t}$

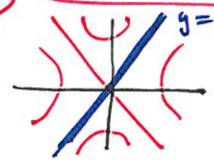
$\lambda = -1 \rightsquigarrow e^{-t}$

Now we find eigenvectors.

$\lambda = 3$: plug into eqns

$$\begin{cases} x + y = 3x \rightsquigarrow y = 2x \\ 4x + y = 3y \rightsquigarrow 4x = 2y \end{cases}$$

SAME LINE



$\Rightarrow$ Want a vector on this line (any vector will work)

– pick $x = 1$
then $y = 2x = 2$ $\Big\} \rightarrow \boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ eigen vector}}$

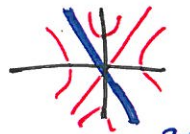
Check: $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 4+2 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \underset{\text{eigenval.}}{\underline{\underline{3}}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = -1$: plug into equations

$$\begin{cases} x + y = -x \rightsquigarrow y = -2x \\ 4x + y = -y \rightsquigarrow 4x = -2y \end{cases}$$

SAME LINE



$\Rightarrow$ Want a vector on this line (any vector will work)

– pick $x = 1$
then $y = -2x = -2$ $\Big\} \rightarrow \boxed{\begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ eigenvector}}$

Check: $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1-2 \\ 4-2 \end{bmatrix}$

$= \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \underset{\text{eigenval}}{\underline{\underline{(-1)}}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Next Time:

More efficient method for writing this calculation.